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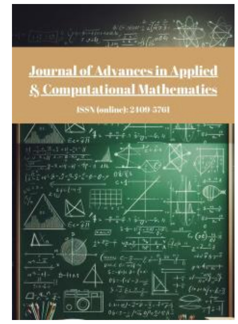

Double Parametric Based Solution of Fuzzy Volterra Integral Equations with Separable Type Kernels

Dennis Xavier[1], Diptiranjan Behera[2,*] and Santhi Dennis Xavier[3]

[1]*School of Mathematics and Statistics, University of Technology, Jamaica, Kingston 6, Jamaica*
[2]*Department of Mathematics, The University of the West Indies, Mona Campus, Kingston 7, Jamaica*
[3]*Department of Mathematics, The Mico University College, Kingston 5, Jamaica*

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ABSTRACT

This paper presents a new approach for solving fuzzy Volterra integral equations with separable type kernels. Here triangular and trapezoidal fuzzy numbers are considered for the analysis. In general, the existing approaches first defuzzify the fuzzy integral equation into a crisp system of integral equations or two different crisp integral equations using the concept of fuzzy arithmetic. Then they solved them to obtain the lower and upper bounds of the fuzzy solution. However, using the proposed technique one has to solve only one crisp integral equation which is obtained by using the concept of double parametric form of fuzzy numbers. This makes the proposed approach more computationally efficient. Laplace Adomian Decomposition Method (LADM) has been implemented here to obtain the solution in double parametric form. The usefulness, and practicality of this method are demonstrated through various examples.

*Corresponding Author
Email: diptiranjanb@gmail.com
Tel: +(1) 876-483-2043

1. Introduction

Fuzzy integral equations hold significant importance in the field of science and engineering particularly in modelling problems where traditional mathematical methods fall short due to uncertainty. These equations are encountered across various fields and applications, including elasticity, plasticity, heat and mass transfer, oscillation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, economics, and medicine.

The principle of integration of fuzzy valued functions was first introduced by Dubois and Prade [1]. Fuzzy integral equations can be categorized into several types such as fuzzy Fredholm integral equations, fuzzy Volterra integral equations, Fuzzy integro-differential equations, fuzzy nonlinear integral equations, and fuzzy integral equations with interval-valued functions and etc. Accordingly, many analytical and numerical methods are proposed to solve fuzzy integral equations. Friedman *et al*. [2] proposed a general procedure for calculating the fuzzy integral for monotonic and single maxima functions. Furthermore, Friedman *et al*. [3] presented an embedding method to solve fuzzy Volterra and Fredholm integral equations. Park *et al*. [4] discussed the existence and uniqueness theorems for the solution of fuzzy Volterra integral equations. However, the Adomian Decomposition Method (ADM) was used to solve ordinary and partial nonlinear differential equations [5]. It has been noticed that it has got much attention in the recent years for solving integral equations such as the Volterra and Fredholm integral equations [6]. Babolian *et al*. [7] applied a numerical procedure for solving fuzzy linear Fredholm integral equation of the second kind using Adomian method. Additionally, Abbasbandy *et al*. [8] suggested a numerical algorithm for solving a linear fuzzy Fredholm integral equation of the second kind by converting the main equation into two crisp linear systems of integral equation.

Furthermore, in Behzadi [9] used ADM and Homotopy Analysis Method (HAM) in solving the fuzzy nonlinear Volterra-Fredholm integral equation of the second kind for the approximate solution. Salahshour *et al.* [10] used fuzzy Laplace transform method to find the solution of fuzzy convolution Volterra integral equation of the second kind with convolution fuzzy and crisp kernel. Narayanamoorthy and Sathiyapriya [11] recommended homotopy perturbation method to evaluate linear and nonlinear fuzzy Volterra integral equations of the second kind. In addition, several authors [12-16] studied integral related work involving fuzzy uncertainty. Mosayebi [17] presented a numerical technique for the solution of fuzzy linear Volterra-Fredholm-Hammerstein integral equations. Ameri and Nezhad [18] proposed based on least squares approximation methods for solving the fuzzy linear Volterra integral equations.

Furthermore, in [19] existence of solution for integro-differential equations has been studied. Differential transform method has been applied in [20] to solve fuzzy Volterra integral equations. Attari and Yazdani [21] solved a fuzzy Volterra-Fredholm integral equations using a computational technique. Additionally, Paripour *et al*. [22] discussed analytic method for solving Abel fuzzy integral equation using Laplace transforms. Uddin and Khan [23] investigated a Volterra integral equation with oscillatory type kernel. Very recently, Khaji *et al*. [24] solved fuzzy Volterra integral equations using composite method and Kapoor *et al*. [25] applied Elzaki transform along with ADM to solve fuzzy linear Volterra integral equation. Fuzzy Volterra integral equations with weakly singular kernels has been solved by Talaei *et al*. [26] using a numerical technique.

This is important to notice that in the above literature the authors defuzzify the fuzzy integral equation into two crisp integral equations to obtain the bounds of the fuzzy solution. To keep this in mind the aim is here to develop a method which can only solve a single integral equation to reduce the computational cost. Accordingly, the proposed method converts the integral equation into a crisp integral equation using the double parametric form of fuzzy numbers.Then it has been solved to obtain the fuzzy solution in double parametric form.

The rest of the paper is organized as follows. Section 2 includes the general idea related to fuzzy set theory. Next Section 3 includes the proposed method. Numerical examples are solved and compared using the proposed method in Section 4. Conclusions are made in Section 5.

2. Preliminaries

In this section, basic definitions and key concepts in fuzzy sets are discussed (Kaufmann and Gupta [27], Zimmermann [28], Ross [29] and Chakraverty *et al*. [30]).

Definition 1. Fuzzy Set

If X is a universe of discourse and x be any element of X, then a fuzzy set $\tilde{A}$ defined on X may be written as a collection of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x): x \in X)\}$, where $\mu_{\tilde{A}}(x)$ is called membership function and $\mu_{\tilde{A}}(x) \in [0, 1]$.

If supremum of the membership function attains 1 that is if $\sup_{x\in X}\{\mu_{\tilde{A}}(x) = 1\}$, then the fuzzy set $\tilde{A}$ is called a normalized fuzzy set.

Definition 2. Fuzzy Number

A fuzzy number $\tilde{A}$ is convex normalized fuzzy set $\tilde{A}$ of the real line R such that

$$\{\mu_{\tilde{A}}(x): R \to [0, 1], \forall\, x \in R\}$$

where, $\mu_{\tilde{A}}$ is called the membership function of the fuzzy set and it is piece-wise continuous.

Definition 3. Triangular Fuzzy Number (TFN)

A triangular fuzzy number $\tilde{A}$ is a convex normalized fuzzy set $\tilde{A}$ of the real line R such that

1. There exists exactly one $x_0 \in R$ with $\mu_{\tilde{A}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{A}$), where $\mu_{\tilde{A}}$ is called the membership function of the fuzzy set.

2. $\mu_{\tilde{A}}(x)$ is piecewise continuous.

Let us consider an arbitrary TFN $\tilde{A} = (a, b, c)$ as shown in Fig. (**1**). The membership function $\mu_{\tilde{A}}$ of $\tilde{A}$ is defined as follows

$$\mu_{\tilde{A}(x)} = \begin{cases} 0, x \le a \\ \dfrac{x-a}{b-a}, a \le x \le b \\ \dfrac{c-x}{c-b}, b \le x \le c \\ 0, x \ge c\,. \end{cases}$$

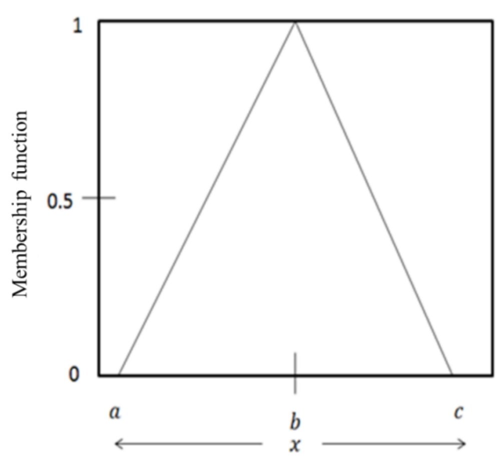

Figure 1: Triangular fuzzy number.

Definition 4. $\alpha-\text{cut}$ or parametric form of TFN.

An arbitrary triangular fuzzy number $\tilde{A}=(a,b,c)$ can be expressed in the $\alpha-\text{cut}$ or parametric form as

$$\tilde{A}=\tilde{A}(\alpha)=\left[\underline{A}(\alpha),\overline{A}(\alpha)\right]=[(b-a)\alpha+a,(b-c)\alpha+c],$$

where $\alpha\in[0,1]$.

Definition 5. Non-negative TFN

A triangular fuzzy number $\tilde{A}=(a,b,c)$ is said to be non-negative if and only if $\mu_{\tilde{A}(x)}=0$ for $a\leq 0$.

Definition 6. Trapezoidal Fuzzy Number (TrFN)

Let us consider an arbitrary trapezoidal fuzzy number $\tilde{A}=(a,b,c,d)$ as shown in Fig. (**2**) and its membership function is defined as

$$\mu_{\tilde{A}(x)}=\begin{cases}0, & x\leq a\\ \dfrac{x-a}{b-a}, & a\leq x\leq b\\ 1, & b\leq x\leq c\\ \dfrac{x-d}{c-d}, & c\leq x\leq d\\ 0, & x\geq d.\end{cases}$$

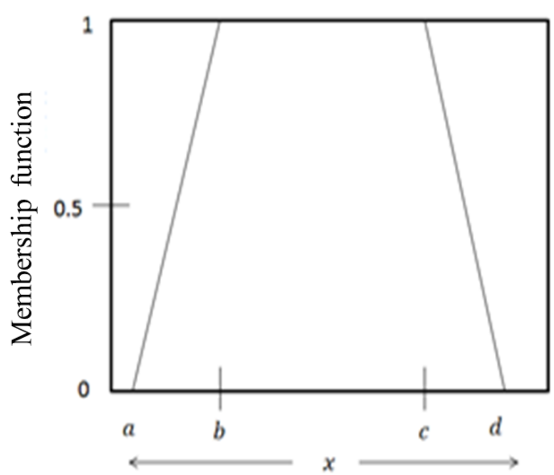

Figure 2: Trapezoidal fuzzy number.

Definition 7. $\alpha-\text{cut}$ form of TrFN

An arbitrary trapezoidal fuzzy number $\tilde{A}=(a,b,c,d)$ can be expressed in the interval form by the $\alpha-\text{cut}$

$$\tilde{A}=\tilde{A}(\alpha)=\left[\underline{A}(\alpha),\overline{A}(\alpha)\right]=[(b-a)\alpha+a,(c-d)\alpha+d],$$

where $\alpha\in[0,1]$

Definition 8. Non-negative TrFN

A trapezoidal fuzzy number $\tilde{A}=(a,b,c,d)$ is said to be non-negative if and only if $\mu_{\tilde{A}(x)}=0$ for $a\leq 0$.

Definition 9. Double parametric form of fuzzy numbers

Let $\widetilde{U}(\alpha) = [\underline{u}\,(\alpha), \bar{u}(\alpha)]$ be the $\alpha-\mathrm{cut}$ or parametric form a TFN or TrFN $\widetilde{U}$. Accordingly, the double parametric form of the above fuzzy number can be defined as

$$\widetilde{U} = \widetilde{U}(\alpha,\beta) = \beta\left[\bar{u}(\alpha) - \underline{u}\,(\alpha)\right] + \underline{u}\,(\alpha),$$

where $\alpha, \beta \in [0,1]$.

Definition 10. Arithmetic properties

Let us consider two fuzzy numbers given by $\tilde{\varphi} = \tilde{\varphi}(\alpha) = [\,\underline{\varphi}(\alpha), \overline{\varphi}(\alpha)]$ and

$\tilde{\delta} = \tilde{\delta}(\alpha) = [\,\underline{\delta}(\alpha), \overline{\delta}(\alpha)]$ in the form of $\alpha-\mathrm{cut}$.

Then fuzzy arithmetic operations can be defined as follows:

i. Addition: $\tilde{\varphi} + \tilde{\delta} = \tilde{\varphi}(\alpha) + \tilde{\delta}(\alpha) = \left[\underline{\varphi}(\alpha) + \underline{\delta}(\alpha), \overline{\varphi}(\alpha) + \overline{\delta}(\alpha)\right]$,

ii. Subtraction: $\tilde{\varphi} - \tilde{\delta} = \tilde{\varphi}(\alpha) - \tilde{\delta}(\alpha) = \left[\underline{\varphi}(\alpha) - \underline{\delta}(\alpha), \overline{\varphi}(\alpha) - \overline{\delta}(\alpha)\right]$,

iii. Multiplication: $\tilde{\varphi} \times \tilde{\delta} = \tilde{\varphi}(\alpha) \times \tilde{\delta}(\alpha) = [\min(\rho), \max(\rho)]$, where

$$\rho = \left[\underline{\varphi}(\alpha) \times \underline{\delta}(\alpha), \overline{\varphi}(\alpha) \times \overline{\delta}(\alpha), \underline{\varphi}(\alpha) \times \underline{\delta}(\alpha), \overline{\varphi}(\alpha) \times \overline{\delta}(\alpha)\right],$$

iv. Division: $\frac{\tilde{\varphi}}{\tilde{\delta}} = \frac{\tilde{\varphi}(\alpha)}{\tilde{\delta}(\alpha)} = [\frac{\underline{\varphi}(\alpha)}{\underline{\delta}(\alpha)}, \frac{\overline{\varphi}(\alpha)}{\overline{\delta}(\alpha)}]$ where, $0 \notin \tilde{\delta}$,

v. Scalar multiplication: for any scalar k and fuzzy number $\tilde{\varphi} = \tilde{\varphi}(\xi) = [\,\underline{\varphi}(\alpha), \overline{\varphi}(\alpha)]$, we have

$$k\tilde{\varphi} = k\tilde{\varphi}(\alpha) = \begin{cases} \left[k\underline{\varphi}(\alpha), \;\; k\overline{\varphi}(\alpha)\right], \text{for } k \geq 0, \\ \left[k\overline{\varphi}(\alpha), \;\; k\underline{\varphi}(\alpha)\right], \text{for } k < 0. \end{cases}$$

Next an important theorem is mentioned, which is essential to understand the proposed method.

Theorem 1. Convolution Theorem (Salahshour *et al*., [20])

If g and h are piece-wise continuous fuzzy-valued functions on $[0,\infty]$ and of exponential order p, then

$$L\{(\tilde{g} * \tilde{h})(t)\} = L\{\tilde{g}(t)\}.L\{(\tilde{h}(t))\} = \tilde{G}(s).\tilde{H}(s), s > p.$$

Proof. Proof of this theorem can be seen in Salahshour *et al*. [20].

3. Proposed Method for Solving Fuzzy Volterra Integral Equations of Separable Type Kernel

Let us consider the classic or crisp form of Volterra integral equation, which is given by

$$v(y) = h(y) + \lambda_1 \int_{\beta_1}^{y} k(y,s)v(s)ds. \tag{1}$$

But, our aim is to investigate the above problem when uncertainties or vagueness are involved in term of fuzzy. Accordingly, to define the fuzzy parameters we have used “~" symbol in the equation. Therefore, now we have the fuzzy form of Eq. (1) as

$$\tilde{v}(y) = \tilde{h}(y) + \lambda_1 \int_{\beta_1}^{y} k(y,s)\tilde{v}(s)ds. \quad (2)$$

Here, $\tilde{v}(y)$is unknown fuzzy function to be determined, $\tilde{h}(y)$ is a known fuzzy valued function, $\lambda_1 \in \mathrm{R}$, $k(y,s)$ is known real valued function and it is said be the kernel of the integral equation, where $y \in [\beta_1, \beta_2]$, $\beta_1, \beta_2 \in \mathrm{R}$ and $\beta_2 < \infty$.

Next considering $\lambda_1 = 1$ and the kernel $k(y,s) = y - s$ which is a separable kernel i.e. $k(y,s) = \sum_{i=0}^{n} h_i(y) g_i(s)$(which satisfies the conditions of Theorem 1 of Park *et al*. [4] and Ullah *et al*. [31] such that Eq. (2) exists and may be unique). Here the assumption has been made that $y, s \in [\beta_1, \beta_2]$ in $k(y,s)$.

And hence Eq. (2) can be expressed as

$$\tilde{v}(y) = \tilde{h}(y) + \int_{\beta_1}^{y} (y - s)\tilde{v}(s)ds.$$

Then, using $\alpha - \mathrm{cut}$, the above equation can be obtained as

$$\tilde{v}(y,\alpha) = \tilde{h}(y,\alpha) + \int_{\beta_1}^{y} (y - s)\tilde{v}(s,\alpha)ds, \quad (3)$$

where, $\tilde{v}(y,\alpha) = [\underline{v}(y,\alpha), \bar{v}(y,\alpha)]$, $\tilde{h}(y,\alpha) = [\underline{h}(y,\alpha), \bar{h}(y,\alpha)]$ and $\tilde{v}(s,\alpha) = [\underline{v}(s,\alpha), \bar{v}(s,\alpha)]$.

Next, using the double parametric form (by Definition 9) for Eq. (3) , we get

$$\tilde{v}(y,\alpha,\beta) = \tilde{h}(y,\alpha,\beta) + \int_{\beta_1}^{y} (y - s)\tilde{v}(s,\alpha,\beta)ds, \quad (4)$$

$\tilde{v}(y,\alpha,\beta) = \beta\left(\bar{v}(y,\alpha) - \underline{v}(y,\alpha)\right) + \underline{v}(y,\alpha)$, $\tilde{h}(y,\alpha,\beta) = \beta\left(\bar{h}(y,\alpha) - \underline{h}(y,\alpha)\right) + \underline{h}(y,\alpha)$ and $\tilde{v}(s,\alpha,\beta) = \beta\left(\bar{v}(s,\alpha) - \underline{v}(s,\alpha)\right) + \underline{v}(s,\alpha)$.

Applying the Laplace transform to Eq. (4), gives

$$L[\tilde{v}(y,\alpha,\beta)] = L[\tilde{h}(y,\alpha,\beta)] + L[\int_{\beta_1}^{y} (y - s)\tilde{v}(s,\alpha,\beta)ds].$$

Next by the convolution theorem the above expression can be reduced to

$$L[\tilde{v}(y,\alpha,\beta)] = L[\tilde{h}(y,\alpha,\beta)] + L[y]L[\tilde{v}(y,\alpha,\beta)].$$

Equivalently it can again be expressed as

$$L[\tilde{v}(y,\alpha,\beta)] = L[\tilde{h}(y,\alpha,\beta)] + \frac{1}{s^2}L[\tilde{v}(y,\alpha,\beta)].$$

Now applying inverse Laplace transform this gives

$$\tilde{v}(y,\alpha,\beta) = \tilde{h}(y,\alpha,\beta)] + L^{-1}\left[\frac{1}{s^2}L[\tilde{v}(y,\alpha,\beta)]\right]. \quad (5)$$

By ADM let us assume the solution of Eq. (5) in a series form as

$$\tilde{v}(y,\alpha,\beta) = \sum_{i=0}^{\infty} \tilde{v}_i(y,\alpha,\beta), \quad (6)$$

where $\tilde{v}_{\mathrm{i}}(y,\alpha,\beta)$ will be determined. Hence, substituting this in Eq. (5) we get

$$\sum_{i=0}^{\infty} \tilde{v}_i\,(y,\alpha,\beta) = \tilde{h}(y,\alpha,\beta) + L^{-1}\left[\frac{1}{s^2} L\left[\sum_{i=0}^{\infty} \tilde{v}_i\,(y,\alpha,\beta)\right]\right]. \quad (7)$$

Expanding the above expression one may have

$$\tilde{v}_0(y,\alpha,\beta) + \tilde{v}_1\,(y,\alpha,\beta) + \tilde{v}_2\,(y,\alpha,\beta) + \cdots = \tilde{h}(y,\alpha,\beta) + L^{-1}\left[\frac{1}{s^2} L[\tilde{v}_0\,(y,\alpha,\beta) + \tilde{v}_1\,(y,\alpha,\beta) + \tilde{v}_2\,(y,\alpha,\beta) + \cdots]\right].$$

Comparing the left and right hand side of the above expression term by term respectively, we get

$$\tilde{v}_0\,(y,\alpha,\beta) = \tilde{h}(y,\alpha,\beta),$$

$$\tilde{v}_1\,(y,\alpha,\beta) = L^{-1}\left[\left(\tfrac{1}{s^2}\right) L[\tilde{v}_0(y,\alpha,\beta)\right],$$

$$\tilde{v}_2\,(y,\alpha,\beta) = L^{-1}\left[\left(\tfrac{1}{s^2}\right) L[\tilde{v}_1(y,\alpha,\beta)\right],$$

$$\tilde{v}_3\,(y,\alpha,\beta) = L^{-1}\left[\left(\tfrac{1}{s^2}\right) L[\tilde{v}_2(y,\alpha,\beta)\right],$$

$$\vdots$$

$$\tilde{v}_{n+1}\,(y,\alpha,\beta) = L^{-1}\left[\left(\tfrac{1}{s^2}\right) L[\tilde{v}_n(y,\alpha,\beta)\right],$$

and so on.

Now substituting these in Eq. (6) one can get the approximate solution of Eq. (4) in double parametric form.

4. Numerical Examples

In this section the aim is to apply the above proposed approach to solve some example problems as below to show the applicability.

Example 1. (Ullah *et al*. [31], Ameri and Nezhad [18])

Let us consider the fuzzy linear Volterra integral equation with $\beta_1 = 0$ and in α −cut or parametric form

$$\tilde{v}(y,\alpha) = \tilde{h}(y,\alpha) + \int_0^y (y-s)\tilde{v}(s)ds, \quad (8)$$

where, $\tilde{h}(y,\alpha) = [3+\alpha, 8-2\alpha], 0 \le y \le 1$.

In Ullah *et al*. [31] the exact solution of this integral equation in parametric form is given as $\tilde{v}\,(y,\alpha) = [(3+\alpha),(8-2\alpha)]\cosh y$.

By the proposed method lets first express Eq. (8) in double parametric form, and accordingly

$$\tilde{v}\,(y,\alpha,\beta) = \tilde{h}\,(y,\alpha,\beta) + \int_0^y (y-s)\tilde{v}\,(s,\alpha,\beta)\,ds. \quad (9)$$

where $\tilde{h}\,(y,\alpha,\beta) = \beta(5-3\alpha) + (3+\alpha)$ and $\beta \in [0,1]$.

Eq. (9) can be then written as

$$\tilde{v}\,(y,\alpha,\beta) = \beta(5-3\alpha) + (3+\alpha) + \int_0^y (y-s)\tilde{v}\,(s,\alpha,\beta)\,ds.$$

Applying the Laplace transform it gives,

$$L[\tilde{v}\,(y,\alpha,\beta)] = L[\beta(5-3\alpha)+(3+\alpha)] + L[\int_0^y (y-s)\tilde{v}\,(s,\alpha,\beta)\,ds].$$

By convolution theorem we have

$$L[\tilde{v}\,(y,\alpha,\beta)] = L[\beta(5-3\alpha)+(3+\alpha)] + L[y]L[\tilde{v}\,(y,\alpha,\beta)].$$

Equivalently now

$$L[\tilde{v}\,(y,\alpha,\beta)] = L[\beta(5-3\alpha)+(3+\alpha)] + \frac{1}{s^2}L[\tilde{v}\,(y,\alpha,\beta)].$$

Applying the inverse Laplace transform we get

$$\tilde{v}\,(y,\alpha,\beta) = \beta(5-3\alpha)+(3+\alpha) + L^{-1}\left[\frac{1}{s^2}L[\tilde{v}\,(y,\alpha,\beta)\right].$$

By ADM let

$$\tilde{v}\,(y,\alpha,\beta) = \sum_{i=0}^{\infty}\tilde{v}_i\,(y,\alpha,\beta)$$

be the solution of the above equation. And accordingly it gives

$$\sum_{i=0}^{\infty}\tilde{v}_i\,(y,\alpha,\beta) = \beta(5-3\alpha)+(3+\alpha) + L^{-1}\left[\left(\frac{1}{s^2}\right)L\left[\sum_{i=0}^{\infty}\tilde{v}_i\,(y,\alpha,\beta)\right]\right].$$

By expanding the series in the above expression we have

$$\begin{aligned}\tilde{v}_0\,(y,\alpha,\beta) + \tilde{v}_1\,(y,\alpha,\beta) + \tilde{v}_2\,(y,\alpha,\beta) + \cdots \\ = \beta(5-3\alpha)+(3+\alpha) + L^{-1}\left[\left(\frac{1}{s^2}\right)L[\tilde{v}_0(y,\alpha,\beta)]\right] + L^{-1}\left[\left(\frac{1}{s^2}\right)L[\tilde{v}_1(y,\alpha,\beta)]\right] + L^{-1}\left[\left(\frac{1}{s^2}\right)L[\tilde{v}_2(y,\alpha,\beta)]\right] \\ + \cdots\end{aligned}$$

Comparing both side term wise we have

$$\tilde{v}_0\,(y,\alpha,\beta) = \beta(5-3\alpha)+(3+\alpha),$$

$$\tilde{v}_1\,(y,\alpha,\beta) = L^{-1}\left[\left(\frac{1}{s^2}\right)L[\tilde{v}_0(y,\alpha,\beta)]\right],$$

$$\tilde{v}_2(y,\alpha,\beta) = L^{-1}\left[\left(\frac{1}{s^2}\right)L[\tilde{v}_1(y,\alpha,\beta)]\right],$$

$$\tilde{v}_3\,(y,\alpha,\beta) = L^{-1}\left[\left(\frac{1}{s^2}\right)L[\tilde{v}_2(y,\alpha,\beta)]\right],$$

and so on.

Simplifying we get

$$\tilde{v}_0\,(y,\alpha,\beta) = \beta(5-3\alpha)+(3+\alpha),$$

$$\tilde{v}_1\,(y,\alpha,\beta) = [\beta(5-3\alpha)+(3+\alpha)]\frac{y^2}{2!},$$

$$\tilde{v}_2\,(y,\alpha,\beta) = [\beta(5-3\alpha)+(3+\alpha)]\frac{y^4}{4!},$$

$$\tilde{v}_3\,(y,\alpha,\beta) = [\beta(5-3\alpha)+(3+\alpha)]\frac{y^6}{6!},$$

and so on.

Substituting these in the assumed series solution and applying the closed form of the series we will get the solution in double parametric form as

$$\tilde{v}\,(y,\alpha,\beta)=[\beta(5-3\alpha)+(3+\alpha)]\cosh y.$$

For $\beta=0$ and 1 one can get the lower and upper bounds of the fuzzy solution in parametric form $(3+\alpha)\cosh y$ and $(8-2\alpha)\cosh y$,respectively. By comparing it can be noticed that it exactly matches with the bounds of exact solution of Ullah *et al*. [31], and Ameri and Nezhad [18].

Example 2. (Ullah *et al*. [31], and Ameri and Nezhad [18])

Here in this example let us consider the fuzzy linear Volterra integral equation with $\beta_1=0$ in parametric form

$$\tilde{v}(y,\alpha)=\tilde{h}(y,\alpha)+\int_0^y (y-s)\tilde{v}(s,\alpha)ds, \tag{10}$$

where $\tilde{h}(y,\alpha)=[\alpha,2-\alpha]\left(1-y-\frac{y^2}{2}\right)$ and $0\le y\le 1$. It is given that (Ullah *et al*. [31]) the exact solution of the above equation is $\tilde{v}\,(y,\alpha)=[\alpha,2-\alpha](1-\sinh y)$.

Now we can apply the proposed method similarly as applied in example 1 and we get the solution in double parametric form as

$$\tilde{v}\,(y,\alpha,\beta)=(\beta((2-\alpha)-\alpha)+\alpha)\left(1-y-\frac{y^2}{2}\right)+$$

$$(\beta((2-\alpha)-\alpha)+\alpha)\left(\frac{y^2}{2!}-\frac{y^3}{3!}-\frac{y^4}{4!}\right)+(\beta((2-\alpha)-\alpha)+\alpha)\left(\frac{y^4}{4!}-\frac{y^5}{5!}-\frac{y^6}{6!}\right)$$
$$+\ (\beta((2-\alpha)-\alpha)+\alpha)\left(\frac{y^6}{6!}-\frac{y^7}{7!}-\frac{y^8}{8!}\right)+\cdots$$

Equivalently it can be expressed as

$$\tilde{v}\,(y,\alpha,\beta)=(\beta((2-\alpha)-\alpha)+\alpha)\left[\left(1-y-\frac{y^2}{2!}\right)+\left(\frac{y^2}{2!}-\frac{y^3}{3!}-\frac{y^4}{4!}\right)+\left(\frac{y^4}{4!}-\frac{y^5}{5!}-\frac{y^6}{6!}\right)+\left(\frac{y^6}{6!}-\frac{y^7}{7!}-\frac{y^8}{8!}\right)+\cdots\right].$$

Next after rearranging the terms we have

$$\tilde{v}\,(y,\alpha,\beta)=(\beta((2-\alpha)-\alpha)+\alpha)\left[1-y-\frac{y^3}{3!}-\frac{y^5}{5!}-\frac{y^7}{7!}+\cdots\right].$$

Finally we have

$$\tilde{v}\,(y,\alpha,\beta)=(\beta((2-\alpha)-\alpha)+\alpha)[1-\sinh y].$$

For $\beta=0$ and 1 one can get the lower and upper bounds of the fuzzy solution in parametric form $\alpha(1-\sinh y)$ and $(2-\alpha)(1-\sinh y)$ respectively. By comparing it can be notices it exactly matches with the bounds of exact solution and Ullah *et al*. [31], and Ameri and Nezhad [18].

Example 3. (Ullah *et al*. [31], and Salahshour and Allahviranloo [20])

In this example we have considered a fuzzy linear Volterra integral equation with $\beta_1=0$ in parametric form

$$\tilde{v}(y,\alpha)=\tilde{h}(y,\alpha)+\int_0^y (y-s)\tilde{v}(s,\alpha)ds, \tag{11}$$

where $\tilde{h}(y,\alpha)=[\alpha-1,1-\alpha]y$, $0\le y\le 1$ and $(y-s)=1$.

It is given in Ullah *et al*. [31] that the exact solution of Eq. (11) is

$$\tilde{v}(y,\alpha)=[(\alpha-1),(1-\alpha)](\sinh y+\cosh y-1).$$

Now we can apply the proposed method similarly as applied for Examples 1 and 2 to obtain the solution in double parametric form as

$$\tilde{v}(y,\alpha,\beta)=[\beta[(1-\alpha)-(\alpha-1)]+(\alpha-1)]y+[\beta[(1-\alpha)-(\alpha-1)]+(\alpha-1)]\frac{y^2}{2!}+[\beta[(1-\alpha)-(\alpha-1)]+(\alpha-1)]\frac{y^3}{3!}+[\beta[(1-\alpha)-(\alpha-1)]+(\alpha-1)]\frac{y^4}{4!}+\cdots.$$

Equivalently then we have now

$$\tilde{v}(y,\alpha,\beta)=[\beta[(1-\alpha)-(\alpha-1)]+(\alpha-1)]\left[y+\frac{y^2}{2!}+\frac{y^3}{3!}+\frac{y^4}{4!}+\cdots\right].$$

Applying the closed form now finally we have

$$\tilde{v}(y,\alpha,\beta)=[\beta[(1-\alpha)-(\alpha-1)]+(\alpha-1)]\,[\sinh y+\cosh y-1].$$

For $\beta=0$ and 1 we get the lower and upper bounds of the fuzzy solution in parametric form as

$$(\alpha-1)(\sinh y+\cosh y-1)\text{ and }(1-\alpha)(\sinh y+\cosh y-1)$$

respectively. By comparing it can be noticed that it exactly matches with the bounds of exact solution of Ullah *et al*. [31], and Salahshour and Allahviranloo [20].

5. Conclusions

In this work, a new method has been successfully proposed to solve Volterra integral equations with separable-type kernels. The methodology incorporates the Laplace Adomian Decomposition Method (LADM) along with the double parametric form of fuzzy numbers.

In general, existing literature converts the original fuzzy integral equation into two crisp integral equations. By solving these equations, one can obtain the bounds of the original solution in parametric form. However, in the proposed approach, only a single equation needs to be solved to derive the final solution in double parametric form. This significantly reduces the computational cost, making the method more efficient.

Various example problems have been tested and compared, demonstrating that the results are in good agreement. It is worth mentioning that while this method performs well when the kernel is positive, it may not be suitable for cases where the kernel is negative.

Future work aims to develop a new method capable of handling problems with negative kernels and extending the present approach to solve other types of fuzzy integral and integro-differential equations, as well as incorporating different types of fuzzy numbers.

Conflict of Interest

The authors declare that they have no conflict of interest.

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Authors' Contributions

All authors have contributed sufficiently in the planning, execution, or analysis of this study to be included as authors. All authors read and approved the final manuscript.

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